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An investigation of the high-field series expansions for the square lattice Ising model

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Abstract. We have used high-field series expansions for the square lattice Ising model to investigate the physical singularity in the magnetisation as a function of the field. High-field series were obtained to order 35 at temperatures $T \approx 0{\cdot}5\,T_c$ and $T \approx 0{\cdot}766\,T_c$ using series expansion techniques based on corner transfer matrices. At neither temperature is there any evidence of a spinodal line: the behaviour is consistent with the predictions of the droplet model, suggesting that the first-order transition line is a line of infinitely differentiable singularities.

1. Introduction

For many years there has been considerable interest in the question of whether the line of first-order transitions in a liquid–gas system corresponds to a line of singularities or whether the properties of one phase can be analytically continued into the two-phase region. Such an analytic continuation could possibly be regarded as representing a metastable state. The present study, like most of the previous investigations, is concerned with the Ising model which can be regarded as a 'lattice gas' model of liquid–gas transitions. The advantages of working with the Ising model are as follows:

(i) It is more tractable than more realistic models.

(ii) The location of the first-order transition line is known.

(iii) Only one phase need be considered because of the symmetry between 'liquid' and 'gas' phases.

Domb (1976) has reviewed the various attempts to study the analytic behaviour of the phase boundary in lattice gas models. Some of the main points are as follows:

(i) Approximate solutions such as the mean-field approximation can be analytically continued into the two-phase region. The continuation is terminated by a spinodal line along which the susceptibility (using 'magnetic' terminology) diverges.

(ii) Essam and Fisher (1963) and Fisher (1967) constructed an approximate 'mimic' partition function for which the line of first-order transitions is a line of singularities.

(iii) Baker (1968) and Gaunt and Baker (1970) have used exact series expansions to search for singular behaviour along the phase boundary. They did not find any indication of singularities on the phase boundary, but they did find an apparent line of singularities inside the two-phase region, as would be expected if properties (i) applied.

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(iv) Domb (1976) has undertaken a detailed analysis of the configurations contributing to the partition function. He suggested that at low temperatures there should be a line of essential singularities in the two-phase region, while at higher temperatures there should be a spinodal line (see figure 1).

More recent work using the renormalisation group (Klein *et al* 1976, Klein 1980) has indicated the existence of a line of essential singularities so that the magnetisation cannot be analytically continued through $H=0$. The two treatments differ, however, in their predictions about the possiblity of analytically continuing $M(H)$ by going around $H=0$ in the complex H plane. A continuation around $H=0$ to negative real H could still be associated with a metastable state (Klein 1980, Fisher 1967).

The present study is based on high-field expansions for the square lattice Ising model: we expand about $\mu=\exp(-2H/kT)=0$, looking for a singularity at $\mu_s \geqslant 1$. The coefficients in the series can be determined exactly, using an algebraic technique which we have described previously in connection with low-temperature series (Baxter and Enting 1979). These low-temperature series have been used by Baker and Kim (1980) to expand the magnetisation about $H=0$ so that a line of singularities would be indicated by a zero radius of convergence. While this latter approach is more direct, it suffers from the disadvantage that the coefficients of the powers of H are not known exactly but are obtained by extrapolating our 23-term low-temperature series.

We conclude that the high-field series are consistent with the predictions of the droplet model rather than indicating the existence of a spinodal.

Section 2 gives some of the relevant technical details concerning the derivation of the series. Section 3 describes the framework within which the series analysis is carried out, showing how the series can be used to distinguish between the different possible types of singularity. Section 4 lists the actual results of these tests, and § 5 indicates the reasons that lead us to interpret these results as arising from a line of infinitely differentiable singularities.

2. Series expansions

The Ising model on the square lattice has a series expansion for Z, the partition function, given by

$$\frac{Z(T,H)}{Z(T=0,H=\infty)}=\kappa(u,\mu)=\sum_{n,m} C_{nm}u^m\mu^n, \tag{2.1}$$

where $u=\exp(-4J/kT)$, $\mu=\exp(-2H/kT)$. The C_{nm} are integers and are non-zero only when $m/2 \leqslant n \leqslant m^2/4$. This means that series (2.1) can be grouped either as a series in u (with coefficients which are polynomials in μ) or as a series in μ (with coefficients which are polynomials in u).

Baxter and Enting (1979) obtained the u series to order u^{23} by using an algebraic technique based on the corner transfer matrix formalism of Baxter (1976). The reduced partition function κ is obtained from a set of equations involving infinite matrices. If an appropriate basis is chosen, then series expansions can be obtained by truncating the matrices at finite size. The technique can also be applied to the μ series. Matrices of dimension 13×13 are sufficient to give κ through to μ^{39}.

The details of the calculation have been changed in several respects:

(i) The equations used by Baxter and Enting involved several products of three or more matrices. In the present calculation additional matrices have been defined so that all products involve only pairs of matrices. In other words, quantities which appeared as temporary intermediate variables in the original formalism are now preserved rather than being recalculated at each stage.

(ii) We work with the temperature fixed (i.e. u is fixed). This means that all calculations involve only series in a single variable. In general, the coefficients occurring in the series for each matrix element would be ratios of polynomials in u, but fixing u reduces the coefficients to rational fractions a/b. The fractions are further simplified by mapping them onto the field of integers modulo p (p prime) via the mapping

$$\begin{aligned} &a \to \tilde{a} && (\tilde{x} = x \bmod p), \\ &b \to \tilde{b}, && \\ &a/b \to (\widetilde{a/b}) = \tilde{a} \otimes (\tilde{b}^{-1}) && (\tilde{x} \otimes \tilde{y} = \tilde{x} \times \tilde{y} \bmod p), \end{aligned}$$

where $\tilde{b}^{-1}$ is defined by $(\tilde{b}^{-1}) \otimes \tilde{b} = 1$.

If bounds are available for numerators and denominators of fractions, then the fractions a/b can be reconstructed from their representatives $(\widetilde{a/b})$ so long as the calculation has been performed for a sufficient number of different primes p. In the series for κ, bounds for the denominators are known because the coefficients are polynomials in u with known degree and integer coefficients.

We have considered the cases of $u = \frac{1}{34}(T/T_c = 0{\cdot}49988\ldots)$ and $u = \frac{1}{10}(T/T_c = 0{\cdot}76555\ldots)$. For a fixed u, the series for κ becomes

$$\kappa(u, \mu) = \sum_{n=0}^{\infty} a_n(u)\mu^n u^{2n}, \tag{2.2}$$

where the coefficients $a_n(u)$ are integers. The coefficients $a_n(\frac{1}{34})$ and $a_n(\frac{1}{10})$ are listed in tables 1 and 2 for $n \leq 35$. Strictly speaking the coefficients are only correct modulo p, where $p = \prod_{i=1}^{m} p_i$, $p_1 = 2^{17} - 99$ and the p_i are consecutive primes. The $u = \frac{1}{34}$ calculation used 19 primes and the $u = \frac{1}{10}$ calculation used 14. Because of the regular behaviour of the series, we do not believe that any additive multiples of p occur.

The procedure of mapping fractions on to integers was described by Borosch and Frankael (1966). The inverses $(\tilde{b}^{-1})$ can be calculated by a modification of Euclid's algorithm for the greatest common divisor (Knuth 1969) or less efficiently by using Fermat's theorem $a^p \equiv a \pmod p$, whence $(\tilde{a}^{-1}) = a^{p-2} \bmod p$ if $a \not\equiv 0 \pmod p$.

3. Possible lines of singularities

Domb (1971) has pointed out that any asymptotic analysis of series coefficients must be based on some assumptions about the possible singularities. For example, most analysis in the theory of critical phenomena is based on the assumption of dominant power-law singularities. In investigating the high-field series we must consider a wider class of singularities. Since widening the class of possible singularities is essentially equivalent to increasing the number of unknown parameters to be estimated, we will only consider the possible forms mentioned by Domb (1976).

Essam and Fisher (1963) and Fisher (1967) used the droplet model to construct a 'mimic' partition function that had a line of singularities at $\mu = 1$.

Baker (1968) and Gaunt and Baker (1970) have used exact series expansions to search for singularities on the phase boundary. They did not find such a singularity, but

Table 1. Series for the partition function coefficients $a_n(u)$, $n = 0$–35, $u = \frac{1}{34}$.

1
1
66
646 8
195 808 8
408 639 561
166 776 730 230
456 702 362 474 82
207 075 315 269 655 03
845 994 756 369 197 391 0
358 975 309 736 915 584 501 8
174 674 255 909 570 267 503 507 8
882 146 758 639 161 618 868 193 742
417 667 336 751 804 678 497 908 537 324
230 023 301 501 341 866 652 528 678 606 146
125 715 425 009 556 023 049 501 391 794 299 826
691 770 310 328 726 115 205 043 429 630 680 217 25
391 779 857 689 171 105 282 868 735 425 155 640 659 36
233 724 485 237 543 576 775 580 402 550 021 681 135 903 70
138 623 639 049 918 271 238 389 456 074 278 002 162 239 044 94
847 308 265 866 920 001 760 862 236 328 830 068 685 447 408 561 8
517 132 854 094 266 351 576 804 326 522 074 195 603 337 882 654 180 1
328 784 621 802 456 744 250 505 197 542 108 547 244 724 130 943 241 840 6
210 074 686 861 034 508 555 568 855 758 618 628 509 812 892 324 348 951 202 8
135 893 705 848 338 045 506 756 675 077 850 357 150 638 094 800 072 847 562 613 5
887 094 032 462 225 069 397 491 800 690 837 000 665 490 306 015 629 383 601 904 749
589 419 504 815 666 906 418 651 875 498 070 205 540 035 547 634 097 336 972 449 044 110
396 710 134 189 835 618 505 086 143 051 639 755 387 765 248 761 040 021 577 070 630 249 764
269 517 268 951 815 214 491 856 959 003 871 399 018 850 019 982 959 660 078 872 486 610 190 104
184 257 111 957 806 136 174 824 318 485 949 709 726 611 948 379 317 757 478 155 335 590 229 036 046
127 728 984 069 437 400 598 664 730 671 177 842 856 555 189 112 713 211 358 181 724 220 310 752 708 494
890 180 846 361 286 370 075 782 481 982 682 553 615 797 451 708 091 020 683 425 154 762 995 252 309 687 42
628 866 313 373 858 746 579 812 822 778 273 099 323 949 410 746 241 096 229 255 634 750 788 103 050 408 742 91
446 745 666 250 606 550 972 806 758 069 260 745 920 274 466 733 856 686 008 784 068 155 128 802 767 994 888 781 95
319 821 343 846 472 535 471 287 218 937 217 299 164 864 261 716 849 602 551 472 401 188 173 294 144 644 442 115 125 94
230 639 859 512 797 878 084 273 687 571 519 665 390 499 170 636 162 964 660 566 861 985 670 413 978 760 337 687 628 654 02

Table 2. Series for the partition function coefficients $a_n(u)$, $n = 0$–35, $u = \frac{1}{10}$.

1
1
18
468
212 40
943 065
530 274 78
285 407 937 0
176 577 539 967
109 467 984 956 22
710 975 088 361 098
475 429 473 995 908 86
326 482 311 261 486 079 8
226 988 834 198 112 518 508
162 134 134 442 239 662 161 46
117 116 411 483 659 519 404 880 2
858 316 981 818 542 855 598 251 01
636 877 319 795 551 113 093 358 732 8
478 383 956 096 199 316 560 660 935 970
362 452 519 183 833 873 010 847 241 751 50
277 420 547 030 596 087 390 344 968 516 238 6
213 955 890 010 279 572 544 957 749 948 234 057
166 395 425 217 420 635 209 957 476 466 504 077 66
130 271 926 217 291 999 840 804 816 785 022 589 538 8
102 678 232 464 874 843 783 160 781 682 017 501 729 111
814 113 027 410 309 599 749 493 330 521 577 728 495 321 3
649 277 221 355 085 372 194 984 857 165 111 156 183 886 382
520 511 707 272 744 435 476 159 975 876 150 425 049 567 437 32
419 380 673 443 845 557 884 098 086 071 806 713 302 852 876 140 0
339 456 599 522 982 436 045 451 376 806 765 999 800 927 449 667 758
275 995 655 105 893 439 856 110 896 032 944 953 845 624 735 050 384 46
225 314 949 357 845 101 023 404 887 000 272 598 627 626 535 150 921 339 8
184 673 878 155 672 274 624 400 925 613 419 326 995 634 930 903 896 379 235
151 919 891 991 593 351 053 546 661 951 439 058 237 876 765 149 762 583 309 47
125 419 256 276 075 067 754 182 694 053 661 903 383 096 868 530 604 017 592 293 0
103 887 485 103 041 752 829 746 120 481 699 637 528 029 219 799 764 794 705 889 226

they did locate what seemed to be a spinodal curve. They assumed that this curve obeyed the scaling equation

$$\tau_s = -D_s(1 - T/T_c)^{\Delta_s}, \qquad \tau = (1-\mu)/(1+\mu), \quad \Delta_s = \tfrac{15}{8},$$

and estimated $D_s = 0{\cdot}39 \pm 0{\cdot}20$. Lines B and B′ on figure 1 show the scaling curve for $D_s = 0{\cdot}19$ and $0{\cdot}59$ respectively.

Domb (1976) suggested a more complicated behaviour in which the spinodal curve existed only for larger temperatures, while for very low temperatures there would be a line of essential singularities *beyond* the first-order transition line $\mu = 1$. Line A on figure 1 shows the limiting (small u) behaviour of the line predicted by Domb: $\mu_s = (1-u)^{-2}$.

This suggestion of Domb's involves representing the asymptotic behaviour of the series coefficients in terms of four unknown parameters, as indicated below. Fortunately this parametrisation turns out to be sufficiently general to include all the other cases that we wish to consider.

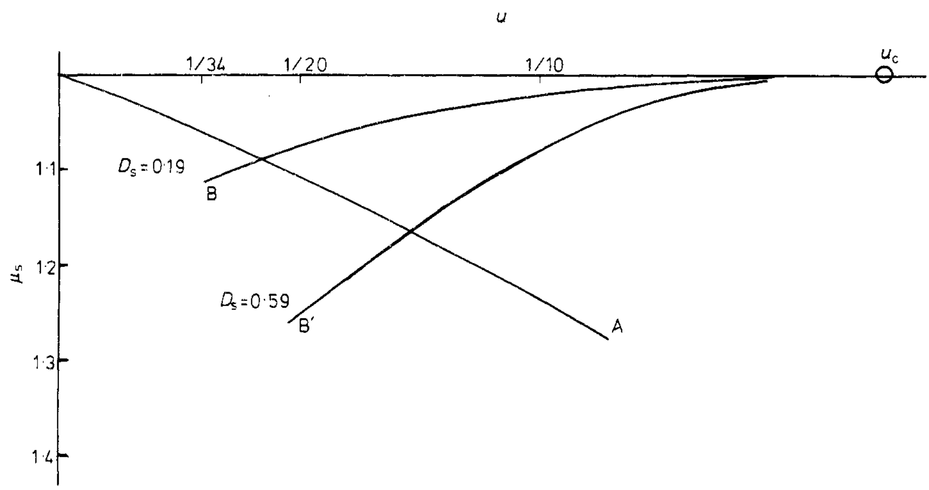

Figure 1. Conjectured lines of singularities in the u–μ plane of the square lattice Ising model. Line A is a line of essential singularities. Lines B and B′ are based on Gaunt and Baker's estimates of the asymptotic behaviour of the apparent spinodal. The two lines define the range of possible positions given by their estimates.

We analyse the series in terms of the asymptotic form

$$b_n \sim x^n/(a^{n^\sigma} n^g), \tag{3.1}$$

with

$$M(\mu) = 1 - 2\sum_n b_n \mu^n. \tag{3.2}$$

Of the four free parameters, x, a, σ and g, x determines the location of the singularity, σ varies only slightly, the possible values all being near $\frac{1}{2}$, and g only becomes important near the critical point. The particular parameter values for the various possibilities described above and sketched in figure 1 are as follows:

(i) Droplet model (Essam and Fisher 1963):

$x = 1$ (singularity at $\mu = 1$),

$a > 1$, $0 < \sigma < 1$ if $T < T_c$ (essential singularity),

$a \to 1$ as $T \to T_c$ (crossover to known critical behaviour),

$\sigma = \frac{8}{15}$, $g = \frac{16}{15}$ near T_c (so as to reproduce the known critical exponents).

(ii) Spinodals ($T < T_c$):

$x < 1$ (singularity at $\mu > 1$),

($a = 1$ or $\sigma = 0$) (no essential singularity),

$g < 2$ (if susceptibility diverges).

(iii) Displaced essential singularity with intersecting spinodal (Domb 1976):

$x < 1$ (singularity at $\mu < 1$),

$\sigma \approx \frac{1}{2}$ (for small u, compact clusters dominate),

$a \to 1$ as $T \to T_0^-$ (crossover to spinodal).

4. Series analysis

As mentioned above, the modified droplet model proposed by Domb involves four unknown parameters. Even with the long series given in the tables we are not in a position to make direct estimates of these parameters.

Our main result is that the series seem to be inconsistent with the existence of a spinodal line. From the previous section it will be seen that a spinodal line places most restrictions on the possible values of the parameters. The 'spinodal' prediction is sufficiently precise for us to be able to test it directly and, as a result of the tests, to reject it as a possiblity. The remaining possibilities that have been proposed are lines of infinitely differentiable singularities. We find that our series are consistent with this type of behaviour with the singularity being near $\mu = 1$. Obviously no series analysis can ever show that the singularity is exactly at $\mu = 1$ (the droplet model) rather than at μ slightly greater than 1 (as suggested by Domb).

Our analysis is based on various forms of the ratio method. The reasons for using the ratio method to the exclusion of other techniques such as Padé approximants are as follows:

(i) The predictions of the droplet model are given most directly in terms of the series coefficients, and the ratio method works directly with the series coefficients.

(ii) The use of Padé approximants requires us to transform our function into some related function which has, at least to a first approximation, a simple pole at the physical singularity. (For power-law singularities the logarithmic derivative of the thermodynamic function is used.) Without some knowledge of the type of singularity it is not possible to make an appropriate choice for the transformation.

(iii) While Padé approximants sometimes display a characteristic pattern of poles and zeros around points which cannot be represented exactly by the approximants, the interpretation of these patterns is, in general, very much a subjective business. Pade analysis of the high-field Ising series has so far proved unfruitful (D Kim, private communication), but it might be hoped that experience with Padé approximants to the Ising model series might ultimately be a guide to the interpretation of Padé analysis at other first-order transitions.

We begin by taking the predictions of the (modified) droplet model for the coefficients

$$b_n \sim x^n/(a^{n^\sigma} n^g). \tag{4.1}$$

The ratios of these coefficients are (see table 3)

$$r_n = b_n/b_{n-1} \sim x(1-g/n)a^{-\sigma n^{\sigma-1}}. \tag{4.2}$$

(The spinodal would correspond to $a^\sigma = 1$ or $r_n = x(1-g/n)$.) The ratios are plotted against $1/n$ in figures 2 and 3. The gradient of the ratio plot is predicted to behave as

$$h_n = \frac{\mathrm{d}}{\mathrm{d}(1/n)} r_n = r_n\left(\frac{-g}{1-g/n} - \sigma(1-\sigma)n^\sigma \ln a\right). \tag{4.3}$$

(For $a^\sigma = 1$ this reduces to $-gx$.)

Table 3. Ratio method analysis of magnetisation series.

	$u=\frac{1}{34}$		$u=\frac{1}{10}$	
n	$r_n=b_n/b_{n-1}$	$-\ln r_n$	$r_n=b_n/b_{n-1}$	$-\ln r_n$
8	0·448 96	0·800 82	0·708 96	0·343 95
9	0·397 56	0·922 41	0·697 71	0·359 95
10	0·407 88	0·896 77	0·722 33	0·325 27
11	0·463 20	0·769 59	0·736 14	0·306 33
12	0·476 66	0·740 96	0·749 66	0·288 14
13	0·443 65	0·812 72	0·753 52	0·283 00
14	0·513 23	0·667 02	0·769 72	0·261 73
15	0·506 57	0·680 10	0·774 24	0·255 88
16	0·507 76	0·677 76	0·782 04	0·245 85
17	0·520 57	0·652 83	0·788 66	0·237 42
18	0·546 48	0·604 25	0·795 60	0·228 26
19	0·541 57	0·613 27	0·799 97	0·223 18
20	0·556 61	0·585 90	0·805 91	0·251 79
21	0·554 36	0·589 93	0·809 98	0·210 74
22	0·576 22	0·551 27	0·814 93	0·204 65
23	0·577 85	0·548 44	0·818 66	0·200 09
24	0·583 93	0·537 97	0·822 61	0·195 27
25	0·588 23	0·530 63	0·826 06	0·191 09
26	0·597 78	0·514 52	0·829 57	0·186 85
27	0·604 63	0·503 13	0·832 24	0·183 15
28	0·609 48	0·495 15	0·835 55	0·179 67
29	0·612 53	0·490 16	0·838 44	0·176 21
30	0·620 36	0·477 46	0·841 19	0·172 93
31	0·622 98	0·473 24	0·843 68	0·169 98
32	0·630 84	0·460 70	0·846 16	0·167 05
33	0·633 74	0·456 11	0·848 44	0·164 36
34	0·638 06	0·449 33	0·850 66	0·161 74
35	0·642 19	0·442 87	0·852 76	0·159 27

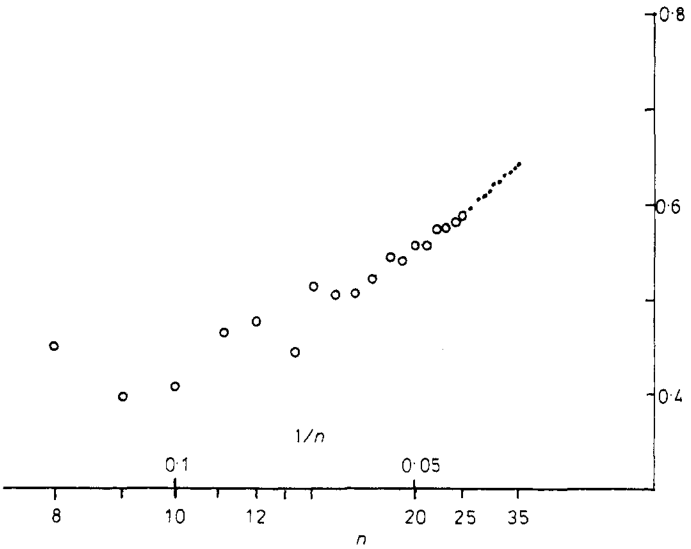

Figure 2. Ratios of the magnetisation series for $u=\frac{1}{34}$, plotted against $1/n$.

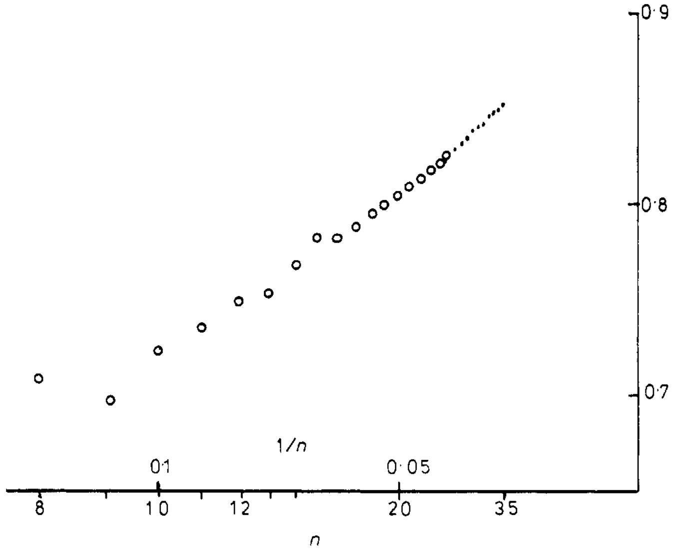

Figure 3. Ratios of the magnetisation series for $u = \frac{1}{10}$, plotted against $1/n$.

Baker (1968) looked at the first 13 ratios (for $T = 0{\cdot}5T_c$) and considered that they indicated a singularity at $\mu \approx 2$.

It will be seen that as more and more terms are considered the curve steepens (g increases) and the intercept moves closer to 1.

The limiting gradient in figure 2 corresponds to $g \approx 6$, implying a singularity $(\mu_s - \mu)^5$ in the magnetisation. This is a much weaker singularity than is generally assumed for a spinodal curve, recalling that we require $g < 2$ if the susceptibility is to diverge.

The large gradient in figure 2 is our main reason for rejecting the possibility of a spinodal curve and moving on to analyse the singularity in terms of the more complicated forms predicted by the droplet model.

Figure 3 shows the ratio plot for $u = \frac{1}{10}$. The change in slope is steady rather than being particularly striking, but the limiting graident $g \approx 2{\cdot}6$ again lies outside possible values for the conventional form of spinodal curve.

In addition to the limiting slopes in the ratio plots being outside the range expected for the singularities of a spinodal curve, the curvature of the lines also indicates a departure from the simple algebraic form of the singularity.

Since the ratio plot has $1/n$ as the ordinate, the larger number of terms implies considerable changes in the spacing of the points so that visual comparisons become unreliable. What we have done is to take sets of m consecutive ratios r_{n+1} to r_{n+m} and used a least-squares fit to calculate gradients. Because of the irregularities in the series we need to take $m \geqslant 6$ before we find the gradient estimates behaving in a smooth manner, but once we use sufficient points we find that between $n = 10$ and $n = 30$ the gradient changes by 50% for $u = \frac{1}{10}$ and by 100% for $u = \frac{1}{34}$. In order to assess the significance of these changes we have applied the same analysis to the high-temperature susceptibility of the honeycomb Ising model for which 32 terms are known (Sykes *et al* 1972). This is one of the few series in which (i) the length is comparable with our high-field series, (ii) the physical singularity is dominant, and (iii) there is a significant amount of irregularity in the ratios, arising from various non-physical singularities. The susceptibility series requires large numbers of points to be fitted before regular gradient estimates are obtained, and shows none of the regular change in gradient estimates that appears in the high-field ratio analysis.

Since many aspects of the ratio analysis point towards a singularity that is more complicated than the power-law singularity expected at a spinodal, we move on to analyse the singularity in terms of the droplet model. We take the logarithms of the ratios (see table 3), which should behave as

$$\ln r_n = \ln x - g/n - \sigma n^{\sigma-1} \ln a. \tag{4.4}$$

While we still have, formally, four unknown parameters, since we need to analyse the series in a manner compatible with Domb's predictions, the problems are fortunately reduced by using our prior knowledge of some of the properties of the function:

(i) The g/n term in (4.4) is the term that gives the crossover to power-law behaviour at the critical point. For low temperatures this term can be ignored as being a small correction, while near the critical point we must use the value $g = \frac{16}{15}$ determined by the known critical exponents. (The fact that g/n is a small correction at low temperatures simplifies our analysis, which is aimed at determining the type of singularity, but it would hinder any attempt to determine whether g varies with temperature.)

(ii) The low-temperature value $\sigma = \frac{1}{2}$ (based on compact droplets) is very close to the critical value $\sigma = \frac{8}{15}$ (obtained from known critical exponents), and the ratio analysis does not depend very greatly on which value of σ is assumed. Again this lack of sensitivity helps our analysis, but would make it difficult to determine the temperature dependence of σ.

For each of the series we plotted $\ln r_n$ and $\ln r_n + \frac{16}{15}n$ against $n^{-\alpha}$ (i.e. $n^{\sigma-1}$) for $\alpha = \frac{1}{2}$ and $\alpha = \frac{7}{15}$ (see figures 4 and 5). For the appropriate choice of α, equation (4.4) predicts that the plots shall be straight lines.

For $u = \frac{1}{34}$ (figure 4) we have the following results:

(i) All of the plots are straight in the sense that the curvature is small compared with the size of the oscillations in the series and thus cannot be detected.

(ii) This means that g/n is indeed a small correction as is expected for low temperatures, and that the analysis cannot give an accurate estimate of the value of σ unless additional assumptions are made.

(iii) The straight-line extrapolations give $H_s = 0{\cdot}0 \pm 0{\cdot}04$ (of course $H_s > 0$ is precluded by the Yang–Lee theorem).

For $u = \frac{1}{10}$ (figure 5):

(i) The plots without the g/n correction extrapolate to values $H_s > 0$ which are not allowed. This indicates that we are in the crossover region in that $\sigma \ln a$ in (4.4) is tending to zero.

(ii) The $\ln r_n + g/n$ plots extrapolate to $H_s \approx -0{\cdot}01$ (using $\alpha = \frac{7}{15}$) or $H_s \approx -0{\cdot}03$ (using $\alpha = \frac{1}{2}$).

5. Conclusions

The conclusions that we draw from the analysis are as follows:

(i) The singularities are too weak to be associated with a spinodal curve.

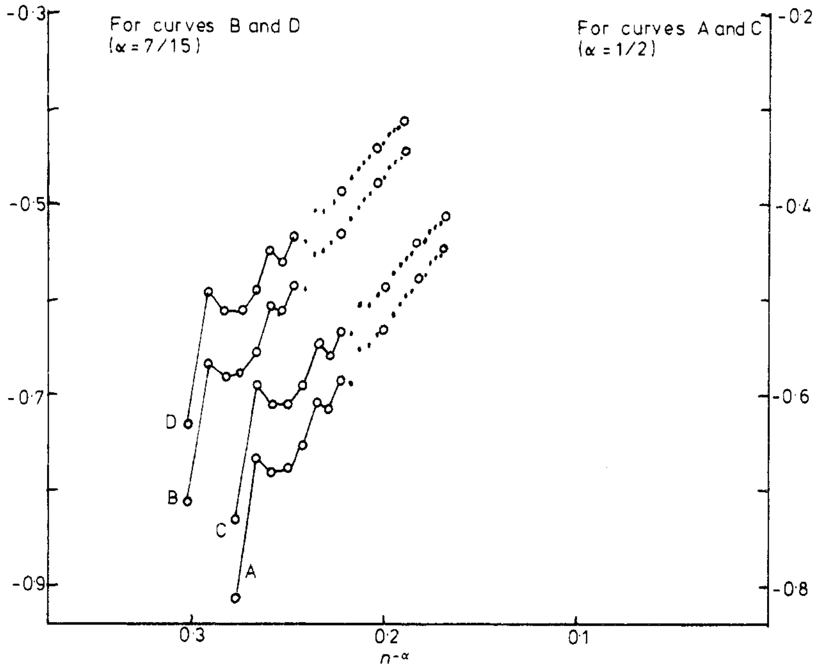

Figure 4. Plot of logarithms against $n^{-\alpha}$ for $u = \frac{1}{34}$: A, $\ln r_n$ against $n^{-1/2}$; B, $\ln r_n$ against $n^{-7/15}$; C, $\ln r_n + \frac{16}{15}n$ against $n^{-1/2}$; D, $\ln r_n + \frac{16}{15}n$ against $n^{-7/15}$.

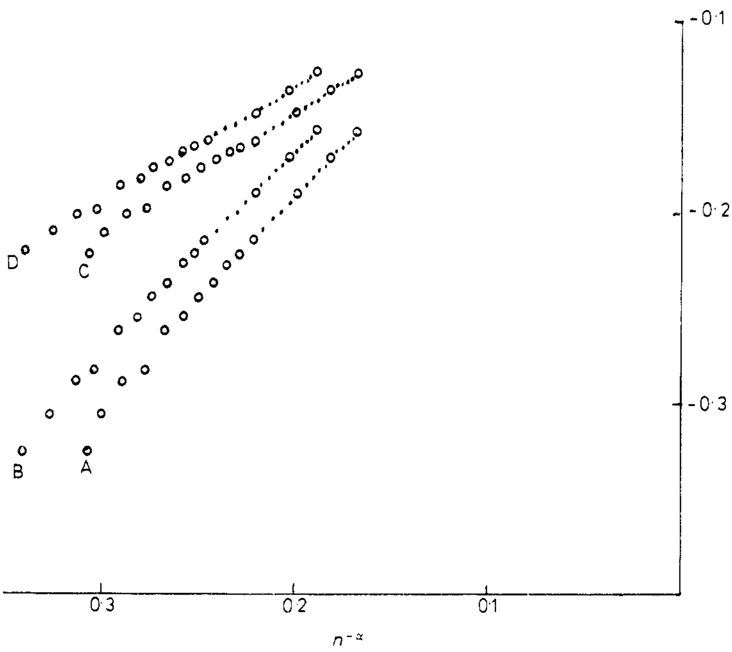

Figure 5. Plot of logarithms of ratios against $n^{-\alpha}$ for $u = \frac{1}{10}$: A, $\ln r_n$ against $n^{-1/2}$; B, $\ln r_n$ against $n^{-7/15}$; C, $\ln r_n + \frac{16}{15}n$ against $n^{-1/2}$; D, $\ln r_n + \frac{16}{15}n$ against $n^{-7/15}$.

(ii) The series are consistent with an infinitely differentiable singularity of the type predicted by the droplet model. As is always the case in series analysis, we can never preclude the possiblity that the singularity has a form that is different from and more complicated than the one we have considered. We have analysed a four-parameter expression of considerable generality. Since we are dealing with a particularly weak singularity, we feel that looking at more general forms is not likely to be fruitful.

(iii) We find no convincing evidence of any displacement of the line of singularities away from $H=0$. Our analysis indicates that any displacement would have to be slightly smaller than predicted by Domb's calculation but, since that calculation was only to leading order, we cannot regard the discrepancy as significant.

(iv) At $u=\frac{1}{10}(T/T_c=0{\cdot}76555\ldots)$ the behaviour of the coefficients is beginning to show a crossover to the power-law behaviour characterising the critical isotherm. The observation of this crossover confirms the necessity of analysing the series in terms of the relatively complicated form (3.1) and of using the ratio method so that the crossover can be incorporated in the framework of the analysis.

(v) While we feel that the most plausible interpretation of our results is that there is a line of infinitely differentiable singularities at $H=0(T<T_c)$ for the square lattice Ising model, there are a number of reasons for suggesting that the behaviour might be different in three dimensions. Firstly, Gaunt and Baker (1970) found that their apparent spinodal was well separated from the first-order transition line. Secondly, the droplet model leads to the prediction $\sigma\approx\frac{8}{13}$ in three dimensions which, as pointed out by Fisher (1967), corresponds to a droplet surface-to-volume ratio which is geometrically impossible. On the other hand, Baker and Kim (1980) found that series expansions about $H=0$ appeared to be divergent in three dimensions as in two. Unfortunately our series expansion techniques cannot be readily extended to three-dimensional systems.

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